CONTINUITY

1. CONTINUOUS FUNCTIONS:
A function for which a small change in the independent variable causes only a small change and not a sudden jump in the dependent variable are called continuous functions. Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.

Continuity of a function at a point:
A function \( f(x) \) is said to be continuous at \( x = a \), if
\[
\lim_{x \to a} f(x) = f(a).
\]
Symbolically, \( f \) is continuous at \( x = a \) if
\[
\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) = f(a), \quad h > 0
\]
i.e. \( (\text{LHL}_{x=a} = \text{RHL}_{x=a}) \) equals value of \( 'f' \) at \( x = a \). It should be noted that continuity of a function at \( x = a \) can be discussed only if the function is defined in the immediate neighbourhood of \( x = a \), not necessarily at \( x = a \).

Ex. Continuity at \( x = 0 \) for the curve can not be discussed.

Illustration 1:
If
\[
f(x) = \begin{cases} \sin \frac{\pi x}{2}, & x < 1 \\ [x], & x \geq 1 \end{cases}
\]
then find whether \( f(x) \) is continuous or not at \( x = 1 \), where \([x]\) denotes greatest integer function.

Solution:
\[
f(x) = \begin{cases} \sin \frac{\pi x}{2}, & x < 1 \\ [x], & x \geq 1 \end{cases}
\]

For continuity at \( x = 1 \), we determine, \( f(1) \), \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^+} f(x) \).

Now, \( f(1) = [1] = 1 \)
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sin \frac{\pi x}{2} = \sin \frac{\pi}{2} = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} [x] = 1
\]

so \( f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \)

\( \therefore \) \( f(x) \) is continuous at \( x = 1 \)

Illustration 2:
Consider
\[
f(x) = \begin{cases} 8^x - 4^x - 2^x + 1 \frac{x^2}{x^2 + \pi x + k/n^4}, & x < 0 \\ e^x \sin x + \pi x + k/n^4, & x \geq 0 \end{cases}
\]
Define the function at \( x = 0 \) if possible, so that \( f(x) \) becomes continuous at \( x = 0 \).

Solution:
\[
f(0^+) = \lim_{h \to 0} \frac{8^h - 4^h - 2^h + 1}{h} = \lim_{h \to 0} \frac{4^h(2^h - 1) - (2^h - 1)}{h^2}
\]
\[
= \lim_{h \to 0} \frac{(4^h - 1)}{h} \frac{(2^h - 1)}{h} = \ell n 4 \cdot \ell n 2
\]

\[
f(0^-) = \lim_{h \to 0} \left(e^x \sin x + \pi x + k/n^4\right) = k/n^4
\]

\( f(x) \) is continuous at \( x = 0 \),

\( \Rightarrow \) \( f(0^+) = f(0^-) = f(0) \) \( \Rightarrow \) \( \ell n 4 \cdot \ell n 2 = k/n^4 \) \( \Rightarrow \) \( k = \ell n 2 \) \( \Rightarrow \) \( f(0) = (\ell n 4)/(\ell n 2) \)
Illustration 3 : Let \( f(x) = \begin{cases} \frac{a(1-x \sin x) + b \cos x + 5}{x^2} & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ \left(1 + \left(\frac{cx + dx^3}{x^2}\right)^{1/x}\right)^{1/x} & \text{if } x > 0 \end{cases} \)

If \( f \) is continuous at \( x = 0 \), then find out the values of \( a \), \( b \), \( c \) and \( d \).

Solution : Since \( f(x) \) is continuous at \( x = 0 \), so at \( x = 0 \), both left and right limits must exist and both must be equal to 3.

Now

\[
\lim_{x \to 0^-} \frac{a(1-x \sin x) + b \cos x + 5}{x^2} = \lim_{x \to 0^+} \frac{(a+b+5) + (-a-b)\frac{x^2}{2} + \ldots}{x^2} = 3 \quad \text{(By the expansions of \( \sin x \) and \( \cos x \))}
\]

If \( \lim_{x \to 0^-} f(x) \) exists then \( a + b + 5 = 0 \) and \( -a - \frac{b}{2} = 3 \) \( \Rightarrow a = -1 \) and \( b = -4 \)

since \( \lim_{x \to 0^+} \left(1 + \left(\frac{cx + dx^3}{x^2}\right)^{1/x}\right)^{1/x} \) exists \( \Rightarrow \lim_{x \to 0^+} \frac{cx + dx^3}{x^2} = 0 \) \( \Rightarrow c = 0 \)

Now \( \lim_{x \to 0^+} (1 + dx)^\frac{1}{x} = \lim_{x \to 0^+} \left(1 + dx\right)^\frac{1}{d} = e^d \)

So \( e^d = 3 \) \( \Rightarrow d = \ln 3 \),

Hence \( a = -1 \), \( b = -4 \), \( c = 0 \) and \( d = \ln 3 \).

Do yourself -1 :

(i) \( f(x) = \begin{cases} \cos x; x \geq 0 \\ x+k; x < 0 \end{cases} \) find the value of \( k \) if \( f(x) \) is continuous at \( x = 0 \).

(ii) \( f(x) = \begin{cases} \frac{|x+2|}{\tan^{-1}(x+2)} & ; x \neq -2 \\ 2 & ; x = -2 \end{cases} \) then discuss the continuity of \( f(x) \) at \( x = -2 \)

2. CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in \((a,b)\) if \( f \) is continuous at each & every point belonging to \((a,b)\).

(b) A function is said to be continuous in a closed interval \([a,b]\) if :

(i) \( f \) is continuous in the open interval \((a,b)\)

(ii) \( f \) is right continuous at ‘a’ i.e. \( \lim_{x \to a^+} f(x) = f(a) \) a finite quantity

(iii) \( f \) is left continuous at ‘b’ i.e. \( \lim_{x \to b^-} f(x) = f(b) \) a finite quantity

Note :

(i) Observe that \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to b^+} f(x) \) do not make sense. As a consequence of this definition, if \( f(x) \) is defined only at one point, it is continuous there, i.e., if the domain of \( f(x) \) is a singleton, \( f(x) \) is a continuous function.

Example : Consider \( f(x) = \sqrt{a-x} + \sqrt{x-a} \).

\( f(x) \) is a singleton function defined only at \( x = a \). Hence \( f(x) \) is a continuous function.

(ii) All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.

(iii) If \( f(x) \) & \( g(x) \) are two functions that are continuous at \( x = c \) then the function defined by :

\[ F_1(x) = f(x) \pm g(x) ; F_2(x) = K f(x) , \text{ where } K \text{ is any real number} ; F_3(x) = f(x) g(x) \text{ are also continuous at } x = c \]

Further, if \( g(c) \) is not zero, then \( F_4(x) = \frac{f(x)}{g(x)} \) is also continuous at \( x = c \).
(iii) Some continuous functions:

<table>
<thead>
<tr>
<th>Function f(x)</th>
<th>Interval in which f(x) is continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant function</td>
<td>(–∞, ∞)</td>
</tr>
<tr>
<td>x^n, n is an integer ≥ 0</td>
<td>(–∞, ∞)</td>
</tr>
<tr>
<td>x^n, n is a positive integer</td>
<td>(–∞, ∞) – {0}</td>
</tr>
<tr>
<td></td>
<td>(–∞, ∞)</td>
</tr>
<tr>
<td></td>
<td>(–∞, ∞) – {0}</td>
</tr>
<tr>
<td>p(x) = a_0x^n + a_1x^{n-1} + ... + a_n</td>
<td>(–∞, ∞)</td>
</tr>
<tr>
<td>p(x)/q(x), where p(x) and q(x) are polynomial in x</td>
<td>(–∞, ∞) – (x : q(x) = 0)</td>
</tr>
<tr>
<td>sin x, cos x, e^x</td>
<td>(–∞, ∞)</td>
</tr>
<tr>
<td>tan x, sec x</td>
<td>(–∞, ∞) – {(2n + 1)π/2 : n ∈ I}</td>
</tr>
<tr>
<td>cot x, cosec x</td>
<td>(–∞, ∞) – {nπ : n ∈ I}</td>
</tr>
<tr>
<td>nx</td>
<td>(0, ∞)</td>
</tr>
</tbody>
</table>

(iv) Some Discontinuous Functions:

<table>
<thead>
<tr>
<th>Functions</th>
<th>Points of discontinuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>[x], {x}</td>
<td>Every Integer</td>
</tr>
<tr>
<td>tan x, sec x</td>
<td>x = ± π/2, ± 3π/2, ...</td>
</tr>
<tr>
<td>cot x, cosec x</td>
<td>x = 0, ± π, ± 2π, ...</td>
</tr>
<tr>
<td>sin 1/x, cos 1/x, 1/x, e^{1/x}</td>
<td>x = 0</td>
</tr>
</tbody>
</table>

Illustration 4: Discuss the continuity of f(x) =

\[
\begin{cases} 
  x + 1, & x < -2 \\
  2x + 3, & -2 \leq x < 0 \\
  x^3 + 3, & 0 \leq x < 3 \\
  x^3 - 15, & x \geq 3 
\end{cases}
\]

Solution: We write f(x) as f(x) =

\[
\begin{cases} 
  -x - 1, & x < -2 \\
  2x + 3, & -2 \leq x < 0 \\
  x^3 + 3, & 0 \leq x < 3 \\
  x^3 - 15, & x \geq 3 
\end{cases}
\]

As we can see, f(x) is defined as a polynomial function in each of intervals (–∞, -2), (-2, 0), (0, 3) and (3, ∞). Therefore, it is continuous in each of these four open intervals. Thus we check the continuity at x = -2, 0, 3.

At the point x = -2

\[
\lim_{x \to -2} f(x) = \lim_{x \to -2} (-x - 1) = -2 - 1 = 1
\]

\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (2x + 3) = 2(-2) + 3 = 1
\]
Therefore,  \( \lim_{x \to -2} f(x) \) does not exist and hence \( f(x) \) is discontinuous at \( x = -2 \).

At the point \( x = 0 \)
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x + 3) = \frac{0^2 + 3}{0^2 + 3} = 3
\]
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 3) = 3
\]
\[f(0) = 0^2 + 3 = 3\]
Therefore \( f(x) \) is continuous at \( x = 0 \).

At the point \( x = 3 \)
\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 + 3) = 3^2 + 3 = 12
\]
\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^3 - 15) = 3^3 - 15 = 12
\]
\[f(3) = 3^3 - 15 = 12\]
Therefore, \( f(x) \) is continuous at \( x = 3 \).

We find that \( f(x) \) is continuous at all points in \( \mathbb{R} \) except at \( x = -2 \).

**Do yourself -2 :**

(i) If \( f(x) = \begin{cases} \frac{x^2}{a} & ; \quad 0 \leq x < 1 \\ -1 & ; \quad 1 \leq x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^2} & ; \quad \sqrt{2} \leq x < \infty \end{cases} \) then find the value of \( a \) & \( b \) if \( f(x) \) is continuous in \( [0, \infty) \).

(ii) Discuss the continuity of \( f(x) = \begin{cases} |x - 3| & ; \quad 0 \leq x < 1 \\ \sin x & ; \quad 1 \leq x \leq \frac{\pi}{2} \text{ in } [0, 3) \\ \log_{\frac{\pi}{2}} x & ; \quad \frac{\pi}{2} < x < 3 \end{cases} \)

3. **REASONS OF DISCONTINUITY :**

(a) Limit does not exist
\[i.e. \quad \lim_{x \to a} f(x) \neq \lim_{x \to a} f(x)\]

(b) \( f(x) \) is not defined at \( x = a \)

(c) \( \lim_{x \to a} f(x) \neq f(a) \)

Geometrically, the graph of the function will exhibit a break at \( x = a \), if the function is discontinuous at \( x = a \). The graph as shown is discontinuous at \( x = 1, 2 \) and \( 3 \).
4. **TYPES OF DISCONTINUITIES**

**Type-1 : (Removable type of discontinuities)** :- In case \( \lim_{x \to a} f(x) \) exists but is not equal to \( f(a) \) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that \( \lim_{x \to a} f(x) = f(a) \) & make it continuous at \( x = a \). Removable type of discontinuity can be further classified as:

(a) **Missing point discontinuity** :

Where \( \lim_{x \to a} f(x) \) exists but \( f(a) \) is not defined.

(b) **Isolated point discontinuity** :

Where \( \lim_{x \to a} f(x) \) exists & \( f(a) \) also exists but; \( \lim_{x \to a} f(x) \neq f(a) \).

**Illustration 5** : Examine the function, \( f(x) = \begin{cases} x - 1, & x < 0 \\ \frac{1}{4}, & x = 0 \\ x^2 - 1, & x > 0 \end{cases} \). Discuss the continuity, and if discontinuous remove the discontinuity by redefining the function (if possible).

**Solution** :

Graph of \( f(x) \) is shown, from graph it is seen that

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = -1, \text{ but } f(0) = \frac{1}{4} \]

Thus, \( f(x) \) has removable discontinuity and \( f(x) \) could be made continuous by taking \( f(0) = -1 \)

\[ \Rightarrow f(x) = \begin{cases} x - 1, & x < 0 \\ -1, & x = 0 \\ x^2 - 1, & x > 0 \end{cases} \]

**Do yourself -3** :

(i) If \( f(x) = \begin{cases} \frac{1}{x - 1}, & 0 \leq x < 2 \\ \frac{x^2 - 3}{5}, & 2 \leq x < 4 \\ 14 - \frac{x^{1/2}}{2}, & x > 4 \end{cases} \), then discuss the types of discontinuity for the function.

**Type-2 : (Non-Removable type of discontinuities)** :-

In case \( \lim_{x \to a} f(x) \) does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

(i) **Finite type discontinuity** : In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(ii) **Infinite type discontinuity** : In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.
(iii) **Oscillatory type discontinuity**:

E.g. \( f(x) = \sin \frac{\pi}{x} \) at \( x = 0 \)

\[ f(x) = \sin \frac{\pi}{x} \]

\[ y = \sin(\pi/x) \]

\( f(x) \) has non-removable oscillatory type discontinuity at \( x = 0 \)

Example: From the adjacent graph note that

(i) \( f \) is continuous at \( x = -1 \)

(ii) \( f \) has isolated discontinuity at \( x = 1 \)

(iii) \( f \) has missing point discontinuity at \( x = 2 \)

(iv) \( f \) has non removable (finite type) discontinuity at the origin.

**Note:** In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at \( x = a \) & LHL at \( x = a \) is called the jump of discontinuity. A function having a finite number of jumps in a given interval \( I \) is called a **piece wise continuous or sectionally continuous** function in this interval.

**Illustration 6:** Show that the function, \( f(x) = \begin{cases} \frac{e^{1/x} - 1}{e^{1/x} + 1} ; & \text{when } x \neq 0 \\ 0 ; & \text{when } x = 0 \end{cases} \) has non-removable discontinuity at \( x = 0 \).

**Solution:** We have, \( f(x) = \begin{cases} \frac{e^{1/x} - 1}{e^{1/x} + 1} ; & \text{when } x \neq 0 \\ 0 ; & \text{when } x = 0 \end{cases} \)

\[ \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0} \frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} = 1 \quad [\because e^{1/h} \to \infty] \]

\[ \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(-1/h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = 0 - \frac{1}{0 + 1} = -1 \quad [\because h \to 0 ; e^{-1/h} \to 0] \]

\[ \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \]. Thus \( f(x) \) has non-removable discontinuity.

**Illustration 7:** \( f(x) = \begin{cases} \cos^{-1} \{ \cot x \} ; & x < \frac{\pi}{2} \\ \pi[x] - 1 ; & x \geq \frac{\pi}{2} \end{cases} \) ; find jump of discontinuity, where \([ ]\) denotes greatest integer & \( \{ \} \) denotes fractional part function.
Solution: \[ f(x) = \begin{cases} \cos^{-1}\{\cot x\} & x < \frac{\pi}{2} \\ \pi[x] - 1 & x \geq \frac{\pi}{2} \end{cases} \]

\[
\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{h \to 0} \cos^{-1}\{\cot x\} = \lim_{h \to 0} \cos^{-1}\left\{\cot\left(\frac{\pi}{2} - h\right)\right\} = \lim_{h \to 0} \cos^{-1}\{\tanh\} = \frac{\pi}{2}
\]

\[
\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^-} \pi[x] - 1 = \lim_{h \to 0} \left[\frac{\pi}{2} + h\right] - 1 = \pi - 1
\]

\[\therefore \text{jump of discontinuity} = \pi - 1 - \frac{\pi}{2} = \frac{\pi}{2} - 1\]

Do yourself -4:

(i) Discuss the type of discontinuity for \( f(x) = \begin{cases} -1 & x \leq -1 \\ |x| & -1 < x < 1 \\ (x + 1) & x \geq 1 \end{cases} \)

5. THE INTERMEDIATE VALUE THEOREM:

Suppose \( f(x) \) is continuous on an interval \( I \), and \( a \) and \( b \) are any two points of \( I \). Then if \( y_0 \) is a number between \( f(a) \) and \( f(b) \), there exists a number \( c \) between \( a \) and \( b \) such that \( f(c) = y_0 \)

Note that a function \( f \) which is continuous in \([a,b]\) possesses the following properties:

(i) If \( f(a) \) & \( f(b) \) posses opposite signs, then there exists atleast one root of the equation \( f(x) = 0 \) in the open interval \((a,b)\).

(ii) If \( K \) is any real number between \( f(a) \) & \( f(b) \), then there exists atleast one root of the equation \( f(x) = K \) in the open interval \((a,b)\).

Note: In above cases the number of roots is always odd.

Illustration 8: Show that the function, \( f(x) = (x - a)^2(x - b)^2 + x \), takes the value \( \frac{a + b}{2} \) for some \( x_0 \in (a, b) \)

Solution: \( f(x) = (x - a)^2(x - b)^2 + x \)

\( f(a) = a \)

\( f(b) = b \)

\[ \frac{a + b}{2} \in (f(a), f(b)) \]

\[ \therefore \text{By intermediate value theorem, there is atleast one } x_0 \in (a, b) \text{ such that } f(x_0) = \frac{a + b}{2}. \]

Illustration 9: Let \( f : [0, 1] \rightarrow [0, 1] \) be a continuous function, then prove that \( f(x) = x \) for atleast one \( x \in [0, 1] \)
Consider \( g(x) = f(x) - x \)

\[
\begin{align*}
g(0) &= f(0) - 0 = f(0) \geq 0 \\
g(1) &= f(1) - 1 \leq 0 \\
\Rightarrow & \quad g(0) \cdot g(1) \leq 0 \\
\Rightarrow & \quad g(x) = 0 \text{ has at least one root in } [0, 1] \\
\Rightarrow & \quad f(x) = x \text{ for at least one } x \in [0, 1]
\end{align*}
\]

**Do yourself - 5:**

(i) If \( f(x) \) is continuous in \([a, b]\) such that \( f(c) = \frac{2f(a) + 3f(b)}{5} \), then prove that \( c \in (a, b) \)

6. **SOME IMPORTANT POINTS:**

(a) If \( f(x) \) is continuous and \( g(x) \) is discontinuous at \( x = a \) then the product function \( \phi(x) = f(x).g(x) \) will not necessarily be discontinuous at \( x = a \), e.g.

\[
f(x) = x \text{ and } g(x) = \begin{cases} 
\sin \left( \frac{\pi}{x} \right) & x \neq 0 \\
0 & x = 0 
\end{cases}
\]

\( f(x) \) is continuous at \( x = 0 \) and \( g(x) \) is discontinuous at \( x = 0 \), but \( f(x).g(x) \) is continuous at \( x = 0 \).

(b) If \( f(x) \) and \( g(x) \) both are discontinuous at \( x = a \) then the product function \( \phi(x) = f(x).g(x) \) is not necessarily be discontinuous at \( x = a \), e.g.

\[
f(x) = -g(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x < 0 
\end{cases}
\]

\( f(x) \) and \( g(x) \) both are discontinuous at \( x = 0 \) but the product function \( f.g(x) \) is still continuous at \( x = 0 \).

(c) If \( f(x) \) and \( g(x) \) both are discontinuous at \( x = a \) then \( f(x) \pm g(x) \) is not necessarily be discontinuous at \( x = a \)

(d) A continuous function whose domain is closed must have a range also in closed interval.

(e) If \( f \) is continuous at \( x = a \) and \( g \) is continuous at \( x = f(a) \) then the composite \( g[f(x)] \) is continuous at \( x = a \), e.g.

\[
f(x) = \frac{x\sin x}{x^2 + 2} \text{ and } g(x) = |x| \text{ are continuous at } x = 0, \text{ hence the composite } (gof)(x) = \frac{x\sin x}{x^2 + 2} \text{ will also be continuous at } x = 0
\]

**Illustration 10:** If \( f(x) = \frac{x + 1}{x - 1} \) and \( g(x) = \frac{1}{x - 2} \), then discuss the continuity of \( f(x) \), \( g(x) \) and \( fog(x) \) in \( \mathbb{R} \).

**Solution:**

\[
f(x) = \frac{x + 1}{x - 1}
\]

\( f(x) \) is a rational function it must be continuous in its domain and \( f \) is not defined at \( x = 1 \).

\[
\therefore \quad f \text{ is discontinuous at } x = 1
\]

\[
g(x) = \frac{1}{x - 2}
\]

\( g(x) \) is also a rational function. It must be continuous in its domain and \( g \) is not defined at \( x = 2 \).

\[
\therefore \quad g \text{ is discontinuous at } x = 2
\]

Now \( fog(x) \) will be discontinuous at \( x = 2 \) (point of discontinuity of \( g(x) \))

Consider \( g(x) = 1 \) \( \text{ (when } g(x) \text{ is point of discontinuity of } f(x) \text{)} \)

\[
\frac{1}{x - 2} = 1 \Rightarrow x = 3
\]

\[
\therefore \quad fog(x) \text{ is discontinuous at } x = 2 \text{ and } x = 3.
\]
Do yourself -6:

(i) Let \( f(x) = \lfloor x \rfloor \) & \( g(x) = \text{sgn}(x) \) (where \( \lfloor . \rfloor \) denotes greatest integer function), then discuss the continuity of

\[
 f(x) \pm g(x), \ f(x)g(x) & \frac{f(x)}{g(x)} \text{ at } x = 0.
\]

(ii) If \( f(x) = \sin|x| \) & \( g(x) = \tan|x| \) then discuss the continuity of \( f(x) \pm g(x) ; \frac{f(x)}{g(x)} \) & \( f(x)g(x) \)

7. SINGLE POINT CONTINUITY:

Functions which are continuous only at one point are said to exhibit single point continuity.

Illustration 11: If \( f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \not\in \mathbb{Q} \end{cases} \), find the points where \( f(x) \) is continuous.

Solution:

Let \( x = a \) be the point at which \( f(x) \) is continuous.

\[
\lim_{{x \to a}} f(x) = \lim_{{x \to -a}} f(x) \Rightarrow a = -a \Rightarrow a = 0 \Rightarrow \text{function is continuous at } x = 0.
\]

Do yourself -7:

(i) If \( g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \not\in \mathbb{Q} \end{cases} \), then find the points where function is continuous.

(ii) If \( f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 1-x^2 & \text{if } x \not\in \mathbb{Q} \end{cases} \), then find the points where function is continuous.

ANSWERS FOR DO YOURSELF

1. (i) 1, (ii) discontinuous at \( x = -2 \)

2. (i) \( a = -1 \) & \( b = 1 \), (ii) Discontinuous at \( x = 1 \) & continuous at \( x = \frac{\pi}{2} \)

3. (i) Missing point removable discontinuity at \( x = 1 \), isolated point removable discontinuity at \( x = 4 \).

4. (i) Finite type non-removable discontinuity at \( x = -1, 1 \)

6. (i) All are discontinuous at \( x = 0 \).

(ii) \( f(x)g(x) \) & \( f(x) \pm g(x) \) are discontinuous at \( x = (2n + 1)\frac{\pi}{2} \); \( n \in \mathbb{I} \)

\[
\frac{f(x)}{g(x)} \text{ is discontinuous at } x = \frac{n\pi}{2}; n \in \mathbb{I}
\]

7. (i) \( x = 0 \), (ii) \( x = \pm \frac{1}{\sqrt{2}} \)
EXERCISE - 01
CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. If \( f(x) = \begin{cases} 
  x + 2, & \text{when } x < 1 \\
  4x - 1, & \text{when } 1 \leq x \leq 3 \\
  x^2 + 5, & \text{when } x > 3 
\end{cases} \), then correct statement is -

(A) \( \lim_{x \to 1} f(x) = \lim_{x \to 3} f(x) \)
(B) \( f(x) \) is continuous at \( x = 3 \)
(C) \( f(x) \) is continuous at \( x = 1 \)
(D) \( f(x) \) is continuous at \( x = 1 \) and \( 3 \)

2. If \( f(x) = \begin{cases} 
  \frac{1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\
  0, & x = 0 
\end{cases} \), then -

(A) \( \lim_{x \to 0^+} f(x) = 1 \)
(B) \( \lim_{x \to 0^-} f(x) = 0 \)
(C) \( f(x) \) is discontinuous at \( x = 0 \)
(D) \( f(x) \) is continuous

3. If function \( f(x) = \frac{\sqrt{1 + x} - \sqrt{1 + x}}{x} \), is continuous function, then \( f(0) \) is equal to -

(A) 2
(B) \( \frac{1}{4} \)
(C) \( \frac{1}{6} \)
(D) \( \frac{1}{3} \)

4. If \( f(x) = \begin{cases} 
  \frac{x^2 - (a + 2)x + 2a}{x - 2}, & x \neq 2 \\
  2, & x = 2 
\end{cases} \) is continuous at \( x = 2 \), then \( a \) is equal to -

(A) 0
(B) 1
(C) -1
(D) 2

5. If \( f(x) = \begin{cases} 
  \frac{\log(1 + 2ax) - \log(1 - bx)}{x}, & x \neq 0 \\
  k, & x = 0 
\end{cases} \), is continuous at \( x = 0 \), then \( k \) is equal to -

(A) \( 2a + b \)
(B) \( 2a - b \)
(C) \( b - 2a \)
(D) \( a + b \)

6. If \( f(x) = \begin{cases} 
  |x| + [-x], & x \neq 2 \\
  \lambda, & x = 2 
\end{cases} \), \( f \) is continuous at \( x = 2 \) then \( \lambda \) is (where \( [\cdot] \) denotes greatest integer) -

(A) -1
(B) 0
(C) 1
(D) 2

7. If \( f(x) = \begin{cases} 
  \frac{1 - \cos 4x}{x^2}, & x < 0 \\
  a, & x = 0 \\
  \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x} - 4}}, & x > 0 
\end{cases} \), then correct statement is -

(A) \( f(x) \) is discontinuous at \( x = 0 \) for any value of \( a \)
(B) \( f(x) \) is continuous at \( x = 0 \) when \( a = 8 \)
(C) \( f(x) \) is continuous at \( x = 0 \) when \( a = 0 \)
(D) none of these
8. Function \( f(x) = \frac{1}{\log |x|} \) is discontinuous at -
   (A) one point                       (B) two points                      (C) three points           (D) infinite number of points

9. Which of the following functions has finite number of points of discontinuity in \( \mathbb{R} \) (where \([ \cdot ]\) denotes greatest integer)
   (A) \( \tan x \)                      (B) \( |x| / x \)                       (C) \( x + [x] \)                     (D) \( \sin [\pi x] \)

10. If \( f(x) = \frac{1 - \tan x}{4x - \pi} \), \( x \neq \pi/4 \) and \( x \in \left[ 0, \frac{\pi}{2} \right] \) is a continuous functions, then \( f(\pi/4) \) is equal to -
    (A) \(-1/2\)                           (B) \(1/2\)                           (C) \(1\)                               (D) \(-1\)

11. The value of \( f(0) \), so that function, \( f(x) = \frac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}} \) becomes continuous for all \( x \), is given by -
    (A) \(a\sqrt{a}\)                     (B) \(-\sqrt{a}\)                     (C) \(\sqrt{a}\)                        (D) \(-a\sqrt{a}\)

12. If \( f(x) = \frac{x - e^x + \cos 2x}{x^2} \), \( x \neq 0 \) is continuous at \( x = 0 \), then -
    (A) \(f(0) = \frac{5}{2}\)          (B) \([f(0)] = -2\)                   (C) \{f(0)\} = -0.5                   (D) \([f(0)].\{f(0)\} = -1.5\)

where \([x]\) and \(\{x\}\) denotes greatest integer and fractional part function.

13. Let \( f(x) = \frac{x(1 + a \cos x) - b \sin x}{x^3} \), \( x \neq 0 \) and \( f(0) = 1 \). The value of \( a \) and \( b \) so that \( f \) is a continuous function are -
    (A) \(5/2, 3/2\)                     (B) \(5/2, -3/2\)                     (C) \(-5/2, -3/2\)                   (D) none of these

14. \( f' \) is a continuous function on the real line. Given that \( x^2 + (f(x) - 2)x - \sqrt{3}.f(x) + 2\sqrt{3} - 3 = 0 \). Then the value of \( f(\sqrt{3}) \) is -
    (A) \(\frac{2(\sqrt{3} - 2)}{\sqrt{3}}\) (B) \(2(1 - \sqrt{3})\)               (C) \(\text{zero}\)                   (D) cannot be determined

**SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)**

15. The value(s) of \( x \) for which \( f(x) = \frac{e^{\sin x}}{4 - \sqrt{x^2 - 9}} \) is continuous, is (are) -
    (A) \(3\)                           (B) \(-3\)                             (C) \(5\)                           (D) all \( x \in (-\infty, -3] \cup [3, \infty) \)

16. Which of the following function(s) not defined at \( x = 0 \) has/have removable discontinuity at the origin ?
    (A) \( f(x) = \frac{1}{1 + 2\cot x} \)           (B) \( f(x) = \cos \left( \frac{\sin x}{x} \right) \)
    (C) \( f(x) = x \sin \frac{\pi}{x} \)              (D) \( f(x) = \frac{1}{\ln |x|} \)
17. Function whose jump (non-negative difference of LHL & RHL) of discontinuity is greater than or equal to one, is/are -

(A) \( f(x) = \begin{cases} \frac{(e^{1/x} + 1)}{x}; & x < 0 \\ \frac{(e^{1/x} - 1)}{(1 - \cos x)}; & x > 0 \end{cases} \)
(B) \( g(x) = \begin{cases} \frac{x^{1/3} - 1}{x^{1/2} - 1}; & x > 1 \\ \frac{\ln x}{(x - 1)}; & \frac{1}{2} < x < 1 \end{cases} \)

(C) \( u(x) = \begin{cases} \arcsin 2x; & x \in \left(0, \frac{1}{2}\right] \\ \arctan 3x; & x < 0 \end{cases} \)

(D) \( v(x) = \begin{cases} \log_3 (x + 2); & x > 2 \\ \log_{1/2} (x^2 + 5); & x < 2 \end{cases} \)

18. If \( f(x) = \frac{1}{x^2 - 17x + 66} \), then \( f\left(\frac{2}{x - 2}\right) \) is discontinuous at \( x = \)

(A) 2  
(B) \( \frac{7}{3} \)  
(C) \( \frac{24}{11} \)  
(D) 6,11

19. Let \( f(x) = [x] \) & \( g(x) = \begin{cases} 0; & x \in Z \\ x^2; & x \in R - Z \end{cases} \), then (where \([.]\) denotes greatest integer function) -

(A) \( \lim_{x \to 1} g(x) \) exists, but \( g(x) \) is not continuous at \( x = 1 \).

(B) \( \lim_{x \to 1} f(x) \) does not exist and \( f(x) \) is not continuous at \( x = 1 \).

(C) \( g \circ f \) is continuous for all \( x \).

(D) \( f \circ g \) is continuous for all \( x \).
SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. Consider the piecewise defined function 
   \[ f(x) = \begin{cases} 
   \sqrt{-x} & \text{if } x < 0 \\
   \frac{1}{x} & \text{if } 0 \leq x \leq 4 \\
   x - 4 & \text{if } x > 4 
   \end{cases} \]
   choose the answer which best describes the continuity of this function -
   (A) the function is unbounded and therefore cannot be continuous
   (B) the function is right continuous at \( x = 0 \)
   (C) the function has a removable discontinuity at 0 and 4, but is continuous on the rest of the real line
   (D) the function is continuous on the entire real line

2. \( f(x) \) is continuous at \( x=0 \), then which of the following are always true ?
   (A) \( \lim_{x \to 0} f(x) = 0 \)
   (B) \( f(x) \) is non-continuous at \( x=1 \)
   (C) \( g(x) = x^2f(x) \) is continuous at \( x = 0 \)
   (D) \( \lim_{x \to 0^+} (f(x) - f(0)) = 0 \)

3. Indicate all correct alternatives if, \( f(x) = \frac{x}{2} - 1 \), then on the interval \([0, \pi]\)
   (A) \( \tan(f(x)) \) & \( \frac{1}{f(x)} \) are both continuous
   (B) \( \tan(f(x)) \) & \( \frac{1}{f(x)} \) are both discontinuous
   (C) \( \tan(f(x)) \) & \( f^{-1}(x) \) are both continuous
   (D) \( \tan(f(x)) \) is continuous but \( \frac{1}{f(x)} \) is not

4. If \( f(x) = \text{sgn}(\cos2x - 2 \sin x + 3) \), where \( \text{sgn}(\ ) \) is the signum function, then \( f(x) \) -
   (A) is continuous over its domain
   (B) has a missing point discontinuity
   (C) has isolated point discontinuity
   (D) has irremovable discontinuity.

5. \( f(x) = \frac{2 \cos x - \sin 2x}{(\pi - 2x)^2} \); \( g(x) = \frac{e^{-\cos x} - 1}{8x - 4\pi} \)
   \( h(x) = f(x) \) for \( x<\pi/2 \)
   \( = g(x) \) for \( x>\pi/2 \)
   then which of the followings does not holds ?
   (A) \( h \) is continuous at \( x = \pi/2 \)
   (B) \( h \) has an irremovable discontinuity at \( x=\pi/2 \)
   (C) \( h \) has a removable discontinuity at \( x = \pi/2 \)
   (D) \( f\left(\frac{\pi^+}{2}\right) = g\left(\frac{\pi^-}{2}\right) \)

6. The number of points where \( f(x) = [\sin x + \cos x] \) (where \([\ ]\) denotes the greatest integer function), \( x \in (0, 2\pi) \)
   is not continuous is -
   (A) 3
   (B) 4
   (C) 5
   (D) 6

7. On the interval \( I = [-2, 2] \), the function \( f(x) = \begin{cases} 
   (x + 1)e^{\frac{1}{[x] + x}} & (x \neq 0) \\
   0 & (x = 0) 
   \end{cases} \)
   then which one of the following hold good ?
   (A) is continuous for all values of \( x \in I \)
   (B) is continuous for \( x \in I \) \(-\{0\} \)
   (C) assumes all intermediate values from \( f(-2) \) & \( f(2) \)
   (D) has a maximum value equal to \( 3/e \)

8. If \( f(x) = \cos\left(\frac{\pi}{x}\right) \cos\left(\frac{\pi}{2}(x - 1)\right) \); where \([\ ]\) is the greatest integer function of \( x \), then \( f(x) \) is continuous at -
   (A) \( x = 0 \)
   (B) \( x = 1 \)
   (C) \( x = 2 \)
   (D) none of these
9. Given \( f(x) = \begin{cases} 3 - \cot^{-1}\left(\frac{2x^3-3}{x^2}\right) & \text{for } x > 0 \\ \{x^2\} \cos\left(e^{1/x}\right) & \text{for } x < 0 \end{cases} \) for \( x > 0 \) where \( \{ \} \) & \( [ ] \) denotes the fractional part and the integral part functions respectively, then which of the following statement does not hold good -

(A) \( f(0^-) = 0 \)

(B) \( f(0^+) = 3 \)

(C) \( f(0) = 0 \Rightarrow \) continuity of \( f \) at \( x = 0 \)

(D) irremovable discontinuity of \( f \) at \( x = 0 \)

10. Let \( 'f' \) be a continuous function on \( \mathbb{R} \). If \( f(1/4^n) = (\sin e^n)e^{-n^2} + \frac{n^2}{n^2+1} \) then \( f(0) \) is -

(A) not unique

(B) 1

(C) data sufficient to find \( f(0) \)

(D) data insufficient to find \( f(0) \)

11. Given \( f(x) = b (\{x\} + [x]) + 1 \) for \( x \geq -1 \)

\( = \sin(\pi(x+a)) \) for \( x < -1 \)

where \([x]\) denotes the integral part of \( x \), then for what values of \( a, b \) the function is continuous at \( x = -1 \)?

(A) \( a = 2n + (3/2); b \in \mathbb{R}; n \in \mathbb{I} \)

(B) \( a = 4n + 2; b \in \mathbb{R}; n \in \mathbb{I} \)

(C) \( a = 4n + (3/2); b \in \mathbb{R}^+; n \in \mathbb{I} \)

(D) \( a = 4n + 1; b \in \mathbb{R}^+; n \in \mathbb{I} \)

12. Consider \( f(x) = \begin{cases} \frac{x\{x\}}{\log_{1-x}2} & \text{for } -1 < x < 0 \\ \frac{\ln(e^x + 2^{\{x\}})}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{cases} \)

where \([\cdot]\) & \((\cdot)\) are the greatest integer function & fractional part function respectively, then -

(A) \( f(0) = \ln2 \Rightarrow f \) is continuous at \( x = 0 \)

(B) \( f(0) = 2 \Rightarrow f \) is continuous at \( x = 0 \)

(C) \( f(0) = e^2 \Rightarrow f \) is continuous at \( x = 0 \)

(D) \( f \) has an irremovable discontinuity at \( x = 0 \)

13. Let \( f(x) = \begin{cases} a \sin^{2n}x & \text{for } x \geq 0 \text{ and } n \to \infty \\ b \cos^{2m}x - 1 & \text{for } x < 0 \text{ and } m \to \infty \end{cases} \) then -

(A) \( f(0^-) \neq f(0^+) \)

(B) \( f(0^-) \neq f(0) \)

(C) \( f(0^-) = f(0) \)

(D) \( f \) is continuous at \( x = 0 \)

14. Consider \( f(x) = \lim_{n \to \infty} \frac{x^n - \sin x^n}{x^n + \sin x^n} \) for \( x > 0, x \neq 1 \) \( f(1)=0 \) then -

(A) \( f \) is continuous at \( x = 1 \)

(B) \( f \) has a finite discontinuity at \( x = 1 \)

(C) \( f \) has an infinite or oscillatory discontinuity at \( x = 1 \)

(D) \( f \) has a removable type of discontinuity at \( x=1 \)

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TRUE / FALSE

1. \( \frac{1}{x + [x]} \) is discontinuous at infinite points. ([ ] denotes greatest integer function).
2. \( \sin |x| + |\sin x| \) is not continuous for all \( x \).
3. If \( f \) is continuous and \( g \) is discontinuous at \( x = a \), then \( f(x)g(x) \) is discontinuous at \( x = a \).
4. There exists a continuous onto function \( f : [0, 1] \rightarrow [0, 10] \), but there exists no continuous onto function \( g : [0, 1] \rightarrow (0, 10) \).
5. If \( f(x) = \frac{\tan(\pi/4 - x)}{\cos 2x} \) for \( x \neq \frac{\pi}{4} \), then the value which can be given to \( f(x) \) at \( x = \frac{\pi}{4} \) so that the function becomes continuous every where in \( (0, \pi/2) \) is \( 1/4 \).
6. The function \( f \), defined by \( f(x) = \frac{1}{1 + 2^{\tan x}} \) is continuous for real \( x \).
7. \( f(x) = \lim_{n \to \infty} \frac{1}{1 + n \sin^2 \frac{x}{\pi}} \) is continuous at \( x = 1 \).
8. If \( f(x) \) is continuous in \( [0, 1] \) and \( f(x) = 1 \) for all rational numbers in \( [0, 1] \) then \( f\left(\frac{1}{\sqrt{2}}\right) = 1 \).

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in Column-I are labelled as A, B, C and D while the statements in Column-II are labelled as p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II.

1. Column-I
   
   (A) If \( f(x) = \begin{cases} \sin\{x\} & x < 1 \\ \cos x + a & x \geq 1 \end{cases} \) where \( \{.\} \) denotes the fractional part function, such that \( f(x) \) is continuous at \( x = 1 \). If \( |k| = \frac{a}{\sqrt{2} \sin \left(\frac{4-\pi}{4}\right)} \) then \( k \) is.
   
   (B) If the function \( f(x) = \frac{1 - \cos(\sin x)}{x^2} \) is continuous at \( x = 0 \), then \( f(0) \) is.
   
   (C) \( f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 1-x & x \not\in \mathbb{Q} \end{cases} \), then the values of \( x \) at which \( f(x) \) is continuous.
   
   (D) If \( f(x) = x + [-x] + [x] \), where \([x]\) and \( \{x\}\) represents integral and fractional part of \( x \), then the values of \( x \) at which \( f(x) \) is discontinuous.

2. Column-II
   
   (p) 1
   
   (q) 0
   
   (r) -1
   
   (s) \( \frac{1}{2} \)
### Assertion & Reason

These questions contain Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true; Statement-II is correct explanation for Statement-I.
(B) Statement-I is true, Statement-II is true; Statement-II is NOT a correct explanation for Statement-I.
(C) Statement-I is true, Statement-II is false.
(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : \( f(x) = \sin x + [x] \) is discontinuous at \( x = 0 \)
   **Because**
   **Statement-II** : If \( g(x) \) is continuous & \( h(x) \) is discontinuous at \( x = a \), then \( g(x) + h(x) \) will necessarily be discontinuous at \( x = a \)
   (A) A  (B) B  (C) C  (D) D

2. Consider \( f(x) = \begin{cases} 
2 \sin(a \cos^{-1} x) & \text{if } x \in (0,1) \\
\sqrt{3} & \text{if } x = 0 \\
a x + b & \text{if } x < 0 
\end{cases} \)
   **Statement-I** : If \( b = \sqrt{3} \) and \( a = \frac{2}{3} \) then \( f(x) \) is continuous in \((\infty, 1)\)
   **Because**
   **Statement-II** : If a function is defined on an interval \( I \) and limit exist at every point of interval \( I \) then function is continuous in \( I \).
   (A) A  (B) B  (C) C  (D) D

3. Let \( f(x) = \begin{cases} 
\frac{\cos x - e^{-x^2/2}}{x^3}, & x \neq 0 \\
0, & x = 0 
\end{cases} \)
   **Statement-I** : \( f(x) \) is continuous at \( x = 0 \).
   **Because**
   **Statement-II** : \( \lim_{x \to 0} \frac{\cos x - e^{-x^2/2}}{x^4} = -\frac{1}{12} \).
   (A) A  (B) B  (C) C  (D) D
4. **Statement-I**: The equation \( \frac{x^3}{4} - \sin \pi x + 3 = \frac{2}{3} \) has at least one solution in \([-2, 2]\)

Because

**Statement-II**: If \( f : [a, b] \to \mathbb{R} \) be a function & let 'c' be a number such that \( f(a) < c < f(b) \), then there is at least one number \( n \in (a, b) \) such that \( f(n) = c \).

(A) A  (B) B  (C) C  (D) D

5. **Statement-I**: Range of \( f(x) = x \left( \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \right) + x^2 + x^4 \) is not \( \mathbb{R} \).

Because

**Statement-II**: Range of a continuous even function can not be \( \mathbb{R} \).

(A) A  (B) B  (C) C  (D) D

6. Let \( f(x) = \begin{cases} 
Ax - B & x \leq -1 \\
2x^2 + 3Ax + B & x \in (-1, 1] \\
4 & x > 1 
\end{cases} \)

**Statement-I**: \( f(x) \) is continuous at all \( x \) if \( A = \frac{3}{4}, B = -\frac{1}{4} \).

Because

**Statement-II**: Polynomial function is always continuous.

(A) A  (B) B  (C) C  (D) D

**COMPREHENSION BASED QUESTIONS**

**Comprehension # 1**

If \( S_n(x) = \frac{x}{x+1} + \frac{x^2}{(x+1)(x^2 + 1)} + \ldots + \frac{x^{2^n}}{(x+1)(x^2 + 1)\ldots(x^{2^n} + 1)} \) and \( x > 1 \)

\[ \lim_{n \to \infty} S_n(x) = \ell \]

\[ g(x) = \begin{cases} 
\sqrt{ax + b} - 1 & x \neq 0 \\
1 & x = 0 
\end{cases} \]

\[ h : \mathbb{R} \to \mathbb{R} \quad h(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 7 \]

On the basis of above information, answer the following questions:

1. If \( g(x) \) is continuous at \( x = 0 \) then \( a + b \) is equal to -
   (A) 0  (B) 1  (C) 2  (D) 3

2. If \( g(x) \) is continuous at \( x = 0 \) then \( g'(0) \) is equal to -
   (A) \( \ell \)  (B) \( \frac{h(6)}{2} \)  (C) \( a - 2b \)  (D) does not exist

3. Identify the incorrect option -
   (A) \( h(x) \) is surjective  (B) domain of \( g(x) \) is \([-1/2, \infty)\)
   (C) \( h(x) \) is bounded  (D) \( \ell = 1 \)
Comprehension # 2

A man leaves his home early in the morning to have a walk. He arrives at a junction of road A & road B as shown in figure. He takes the following steps in later journey:
(a) 1 km in north direction
(b) changes direction & moves in north-east direction for $2\sqrt{2}$ k.ms.
(c) changes direction & moves southwards for distance of 2 km.
(d) finally he changes the direction & moves in south-east direction to reach road A again.

Visible/Invisible path :- The path traced by the man in the direction parallel to road A & road B is called invisible path, the remaining path traced is visible.

Visible points :- The points about which the man changes direction are called visible points except the point from where he changes direction last time

Now if road A & road B are taken as x-axis & y-axis then visible path & visible point represents the graph of $y = f(x)$.

On the basis of above information, answer the following questions:

1. The value of x at which the function is discontinuous -
   (A) 2  (B) 0  (C) 1  (D) 3

2. The value of x at which $f(f(x))$ is discontinuous -
   (A) 0  (B) 1  (C) 2  (D) 3

3. If $f(x)$ is periodic with period 3, then $f(19)$ is -
   (A) 2  (B) 3  (C) 19  (D) none of these

MISCELLANEOUS TYPE QUESTION

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<tr>
<th>MISCELLANEOUS TYPE QUESTION</th>
<th>ANSWER KEY</th>
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<tr>
<td>True / False</td>
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<tr>
<td>Match the Column</td>
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<tr>
<td>1. (A) -&gt; (p, r); (B) -&gt; (s); (C) -&gt; (s); (D) -&gt; (p, q, r) 2. (A) -&gt; (q, s); (B) -&gt; (p, r, s); (C) -&gt; (q); (D) -&gt; (q)</td>
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<td>Assertion &amp; Reason</td>
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EXERCISE - 4 [A]  

CONCEPTUAL SUBJECTIVE EXERCISE

1. If \( f(x) = \begin{cases} 
-x^2, & \text{when } x \leq 0 \\
5x - 4, & \text{when } 0 < x \leq 1 \\
4x^2 - 3x, & \text{when } 1 < x < 2 \\
x + 4, & \text{when } x \geq 2 
\end{cases} \), discuss the continuity of \( f(x) \) in \( \mathbb{R} \).

2. Let \( f(x) = \begin{cases} 
2\sin x & \text{for } -\pi \leq x \leq -\frac{\pi}{2} \\
ax + b & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\
\cos x & \text{for } \frac{\pi}{2} \leq x \leq \pi 
\end{cases} \). If \( f \) is continuous on \([-\pi, \pi]\) then find the values of \( a \) & \( b \).

3. Determine the values of \( a, b \) & \( c \) for which the function
\[
 f(x) = \begin{cases} 
\frac{\sin (a+1)x + \sin x}{x} & \text{for } x < 0 \\
c & \text{for } x = 0 \\
\frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for } x > 0 
\end{cases}
\]
is continuous at \( x = 0 \).

4. Determine the kind of discontinuity of the function \( y = -\frac{2^{1/x} - 1}{2^{1/x} + 1} \) at the point \( x = 0 \).

5. Suppose that \( f(x) = x^3 - 3x^2 - 4x + 12 \) and \( h(x) = \begin{cases} 
f(x) & \text{for } x \neq 3 \\
K & \text{for } x = 3 
\end{cases} \) then

(a) find all zeros of \( f \)
(b) find the value of \( K \) that makes \( h \) continuous at \( x = 3 \)
(c) using the value of \( K \) found in (b) determine whether \( h \) is an even function.

6. Draw the graph of the function \( f(x) = x - |x-x^2| \), \( -1 \leq x \leq 1 \) & discuss the continuity or discontinuity of \( f \) in the interval \(-1 \leq x \leq 1 \).

7. If \( f(x) = \frac{\sin 3x + A \sin 2x + B \sin x}{x^5} \) \((x \neq 0)\) is continuous at \( x = 0 \), then find \( A \) & \( B \). Also find \( f(0) \).

8. (a) Let \( f(x + y) = f(x) + f(y) \) for all \( x, y \) & if the function \( f(x) \) is continuous at \( x = 0 \), then show that \( f(x) \) is continuous at all \( x \).
(b) If \( f(x \cdot y) = f(x) \cdot f(y) \) for all \( x, y \) and \( f(x) \) is continuous at \( x = 1 \). Prove that \( f(x) \) is continuous for all \( x \) except at \( x = 0 \). Given \( f(1) \neq 0 \).

9. Examine the continuity at \( x = 0 \) of the sum function of the infinite series:
\[
x + \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots \ldots \ldots \infty
\]

10. Show that:
(a) a polynomial of an odd degree has at least one real root
(b) a polynomial of an even degree has at least two real roots if it attains at least one value opposite in sign to the coefficient of its highest-degree term.

<table>
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<tr>
<th>CONCEPTUAL SUBJECTIVE EXERCISE</th>
<th>ANSWER KEY</th>
<th>EXERCISE-4(A)</th>
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<tbody>
<tr>
<td>1. continuous everywhere except at ( x = 0 )</td>
<td>2. ( a = -1 ), ( b = 1 )</td>
<td></td>
</tr>
<tr>
<td>3. ( a = -3/2 ), ( b \neq 0 ), ( c = 1/2 )</td>
<td>4. non-removable - finite type</td>
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<tr>
<td>5. (a) (-2, 2, 3) (b) ( K = 5 ) (c) even</td>
<td>6. ( f ) is continuous in (-1 \leq x \leq 1 )</td>
<td></td>
</tr>
<tr>
<td>7. ( A = -4 ), ( B = 5 ), ( f(0) = 1 )</td>
<td>9. discontinuous at ( x = 0 )</td>
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</table>
1. Given \( f(x) = \sum_{r=1}^{n} \tan \left( \frac{x}{2^r} \right) \sec \left( \frac{x}{2^{r-1}} \right) \); \( r, n \in \mathbb{N} \)

\[
g(x) = \begin{cases} 
\ln \left( f(x) + \tan \frac{x}{2^n} \right) - \left( f(x) + \tan \frac{x}{2^{n+1}} \right)^n \left( \sin \left( \tan \frac{x}{2} \right) \right) & ; \quad x \neq \pi / 4 \\
K & ; \quad x = \pi / 4
\end{cases}
\]

where \([ ]\) denotes the greatest integer function and the domain of \(g(x)\) is \((0, \pi / 2)\). Find the value of \(k\), if possible, so that \(g(x)\) is continuous at \(x = \pi / 4\). Also state the points of discontinuity of \(g(x)\) in \((0, \pi / 4)\), if any.

2. Let \(f(x)=\begin{cases} 1 + x^2, & x < 0 \\
x^2 - 1, & x \geq 0 \end{cases} \quad g(x)=\begin{cases} (x-1)^{1/3}, & x < 0 \\
(x+1)^{1/2}, & x \geq 0 \end{cases}
\)

Discuss the continuity of \(g(f(x))\).

3. Discuss the continuity of \(f\) in \([0,2]\) where \(f(x)=\begin{cases} 4x - 5[x] & \text{for } x > 1 \\
[\cos \pi x] & \text{for } x \leq 1 \end{cases}\); where \([x]\) is the greatest integer not greater than \(x\). Also draw the graph.

4. Discuss the continuity of the function \(f(x) = \lim_{n \to \infty} \frac{\ln(2 + x) - x^{2n} \sin x}{1 + x^{2n}}\) at \(x = 1\)

5. Consider the function \(g(x)=\begin{cases} 1 - a^x + x a^x \tan a & \text{for } x < 0 \\
\frac{a^x - x \ln 2 - x \tan a - 1}{x^2} & \text{for } x > 0 \end{cases}\) where \(a > 0\).

Find the value of ‘a’ & ‘\(g(0)\)’ so that the function \(g(x)\) is continuous at \(x = 0\).

6. Let \(f(x) = \begin{cases} \frac{\pi}{2} - \sin^{-1}(1 - (\{x\})^2) \cdot \sin^{-1}(1 - \{x\}) & \text{for } x \neq 0 \\
\frac{\pi}{2} \left( \{x\} - (\{x\})^3 \right) & \text{for } x = 0 \end{cases}\)

Consider another function \(g(x);\) such that

\[
g(x) = \begin{cases} f(x) & \text{for } x \geq 0 \\
2\sqrt{2} f(x) & \text{for } x < 0 \end{cases}
\]

Discuss the continuity of the functions \(f(x)\) & \(g(x)\) at \(x = 0\).

7. \(f(x)=\begin{cases} \frac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x} & \text{for } x > 0 \\
\frac{\ln(1 + x + x^2) + \ln(1 - x + x^2)}{\sec x - \cos x} & \text{for } x < 0 \end{cases}\)

If ‘\(f\)’ is continuous at \(x = 0\), find ‘\(a\)’

now if \(g(x)=\begin{cases} \ln \left( \frac{2 - x}{a} \right) \tan(x - a) & \text{for } x \neq a, a \neq 0, a > 0 \end{cases}\)

If ‘\(g\)’ is continuous at \(x = a\) then show that \(g(e^a) = -e\)
8. Let \([x]\) denote the greatest integer function & \(f(x)\) be defined in a neighbourhood of 2 by

\[
f(x) = \begin{cases} 
\frac{\exp\left\{(x + 2) \tan(x)\right\}}{4 - 16} & , \ x < 2 \\
\frac{1 - \cos(x - 2)}{(x - 2) \tan(x - 2)} & , \ x > 2 
\end{cases}
\]

Find the value of \(A\) & \(f(2)\) in order that \(f(x)\) may be continuous at \(x = 2\).

9. If \(g : [a, b] \rightarrow [a, b]\) is continuous show that there is some \(c \in [a, b]\) such that \(g(c) = c\).

10. Let \(y_n(x) = x^2 + \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} + \cdots + \frac{x^2}{(1 + x^2)^{n-1}}\) and \(y(x) = \lim_{n \to \infty} y_n(x)\). Discuss the continuity of \(y_n(x) (n = 1, 2, 3, \ldots)\) and \(y(x)\) at \(x = 0\).
1. If \( f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \not\in \mathbb{Q} \end{cases} \), then \( f \) is continuous at-

   (1) Only at zero  
   (2) only at 0, 1  
   (3) all real numbers  
   (4) all rational numbers  

[AIEEE 2002]

2. If \( f(x) = \begin{cases} xe^{\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{cases} \), then \( f(x) \) is-

   (1) discontinuous everywhere  
   (2) continuous as well as differentiable for all \( x \)  
   (3) continuous for all \( x \) but not differentiable at \( x = 0 \)  
   (4) neither differentiable nor continuous at \( x = 0 \)  

[AIEEE 2003]

3. Let \( f(x) = \frac{1 - \tan x}{4x - \pi} \), \( x \neq \frac{\pi}{4}, x \in \left[0, \frac{\pi}{2}\right] \). If \( f(x) \) is continuous in \( \left[0, \frac{\pi}{2}\right] \), then \( f \left(\frac{\pi}{4}\right) \) is-

   (1) 1  
   (2) \( \frac{1}{2} \)  
   (3) \( -\frac{1}{2} \)  
   (4) \(-1\)  

[AIEEE 2004]

4. The function \( f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) given by \( f(x) = \frac{1}{x} - \frac{2}{e^x - 1} \) can be made continuous at \( x = 0 \) by defining \( f(0) \) as-

   (1) 2  
   (2) \( -1 \)  
   (3) 0  
   (4) 1  

[AIEEE 2007]

5. The values of \( p \) and \( q \) for which the function \( f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , x < 0 \\ \frac{q}{\sqrt{x} + x^2 - \sqrt{x}} & , x > 0 \end{cases} \) is continuous for all \( x \) in \( \mathbb{R} \), are:-

   (1) \( p = -\frac{3}{2}, q = \frac{1}{2} \)  
   (2) \( p = \frac{1}{2}, q = \frac{3}{2} \)  
   (3) \( p = \frac{1}{2}, q = -\frac{3}{2} \)  
   (4) \( p = \frac{5}{2}, q = \frac{1}{2} \)  

[AIEEE 2011]

6. Define \( F(x) \) as the product of two real functions \( f_1(x) = x, x \in \mathbb{R}, \) and \( f_2(x) = \begin{cases} \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \) as follows:

\[
F(x) = \begin{cases} f_1(x)f_2(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
\]

[AIEEE 2011]

Statement-1 : \( F(x) \) is continuous on \( \mathbb{R} \).
Statement-2 : \( f_1(x) \) and \( f_2(x) \) are continuous on \( \mathbb{R} \).

(1) Statement-1 is false, statement-2 is true.
(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.
(3) Statement-1 is true, statement-2 is true, statement-2 is not a correct explanation for statement-1.
(4) Statement-1 is true, statement-2 is false.

7. Consider the function, \( f(x) = |x - 2| + |x - 5|, x \in \mathbb{R} \).

Statement-1 : \( f(4) = 0 \).
Statement-2 : \( f \) is continuous in \( [2, 5] \), differentiable in \( (2, 5) \) and \( f(2) = f(5) \).

(1) Statement-1 is true, Statement-2 is false.
(2) Statement-1 is false, Statement-2 is true.
(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(4) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
1. Discuss the continuity of the function \( f(x) = \begin{cases} \frac{e^{1/(x-1)} - 2}{e^{1/(x-1)} + 2}, & x \neq 1 \\ 1, & x = 1 \end{cases} \) at \( x = 1 \).

[REE 2001 (Mains), 3]

2. For every integer \( n \), let \( a_n \) and \( b_n \) be real numbers. Let function \( f : \mathbb{R} \to \mathbb{R} \) be given by

\[
f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n), \end{cases}
\]

for all integers \( n \).

If \( f \) is continuous, then which of the following holds(s) for all \( n \)?

(A) \( a_{n-1} - b_{n-1} = 0 \)  
(B) \( a_n - b_n = 1 \)  
(C) \( a_n - b_{n+1} = 1 \)  
(D) \( a_{n-1} - b_n = -1 \)

[JEE 2012, 4]